The Infinite Newtonian Universe

Version 1
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Introduction

This post concerns the derivations of an equation of motion for matter in an infinite, uniform, Newtonian universe. The derivations are presented in a couple of serious academic lectures available on the internet. Both lecture say that Newton's assumption that in his universe the forces of gravity would be symmetric, so there would be no net gravitational force on any of the matter was in fact false.

The first derivation, which I shall call the shell approach, is part of a course in cosmology presented by Leonard Susskind at the Stanford University. The second video is presented by Alan Guth of M.I.T. In this lecture he rejects the shell approach for reasons similar to mine, although he uses a slightly different approach and and also does not cover a crucial part of Susskind justification for the approach. He also presents two alternative approaches to the derivation based on Gauss's Law and the Laplacian – both of which I also find highly suspect.

The Suskind Lecture is at :-

https://www.youtube.com/watch?v=P-medYaqVak

with the key section between timestamps 0:30:00 and 0:49:00.

The Guth Lecture is at :-

https://ocw.mit.edu/courses/physics/8-286-the-early-universe-fall-2013/video-lectures/lecture-5-cosmological-redshift-and-the-dynamics-of-homogeneous-expansion/

With the key sections between timestamps 0:50:00 and 0:59:00.

I would recommend that you watch these sections before reading this document.

The Shell Approach

This is the approach used by Susskind in his lecture at timestamp 0:30:00. The idea it to start with an infinite universe with a uniform density. This density only needs to be uniform on the large scale, but to make things clearer I will treat it as uniform at the smallest scale.

Figure 1 - Uniform Density of Matter

Within this space we chose an arbitrary reference point for the observer, 'O', with coordinate system centred on himself.

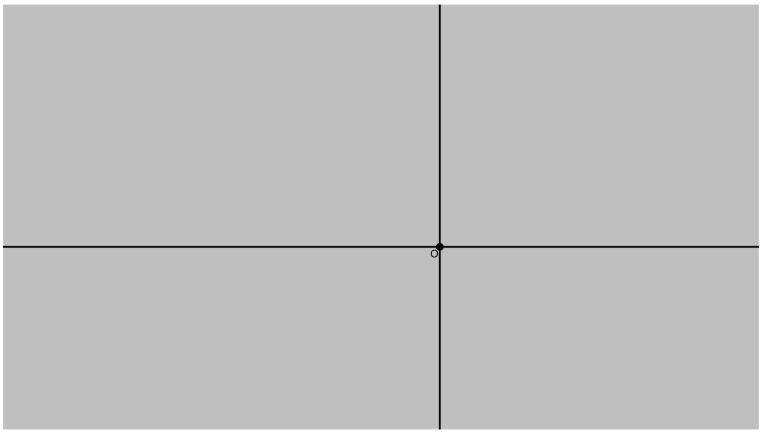


Figure 2: - Observer and his Coordinates

In the derivation in the Susskind video the coordinates are in terms of a scale factor 'a'. This is an unnecessary complication so we will stick to actual distances.

We will now take a target region of space that is small compared to its distance, 'R', from the observer. We will label this target 'T'.

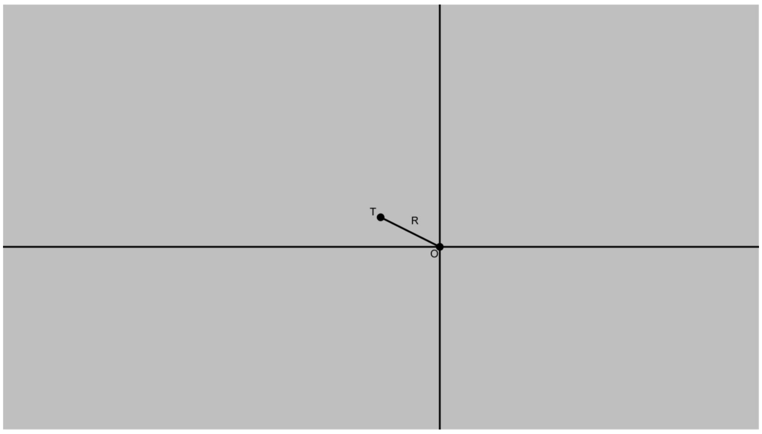


Figure 3 - Target Mass

In order to calculate the gravitational effect of the rest of the universe on the target mass we will divide the universe into a number of regions. The first region will be a large sphere centred on the observer and with a radius equal to the distance from the observer to the target mass, 'R'.

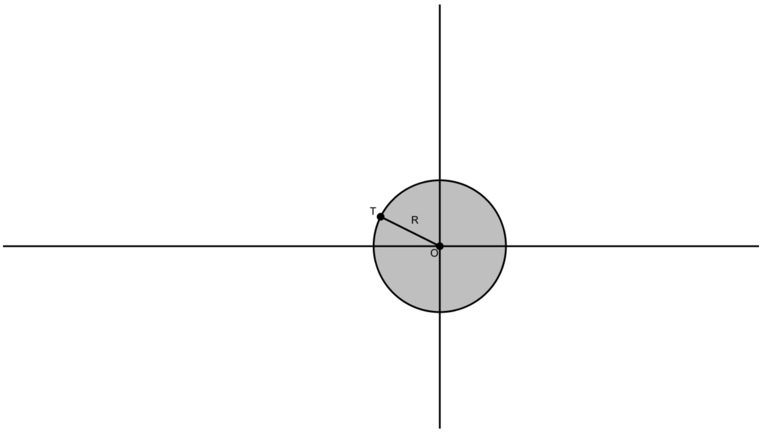


Figure 4 - Mass within Radius R

The gravitational force '*F*' of this sphere on the test mass is given by the standard Newtonian formula for a mass on the surface of a sphere :-

$$F = \frac{GMm}{R^2}$$ Equation 1

This force will always be directly towards the observer.

The mass of the sphere, '*M*', will be given by the volume times the density:-

$$M = \frac{4}{3}\pi R^3 \rho$$ Equation 2

The test mass can be derived in the same way, but as mass of the test matter will remain constant we will leave the mass as 'm' (it will be divided out soon anyway).

So the force on the test mass will be :-

$$F = \frac{G\frac{4}{3}\pi R^3 \rho m}{R^2} \qquad \text{Equation 3}$$

Which will reduce to :-

$$F = \frac{4\pi G R \rho m}{3} \qquad \text{Equation 4}$$

The mass for the rest of the universe will be divided up into a series of concentric shells centred on the observer. If we take one such shell as shown :-

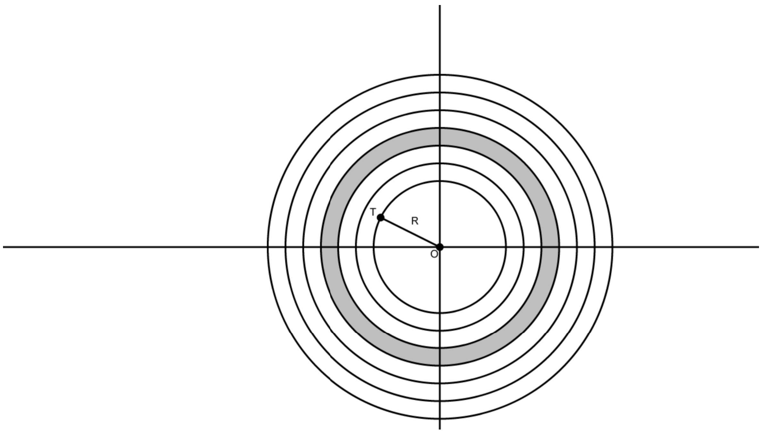


Figure 5: Shell of Matter

The gravitational force on any point within a hollow sphere is actually zero. I will not prove this here but you can find proofs on-line. So if we sum up all the shells outside the radius R we will get nothing. So the only gravitational force on the target mass will be from the central sphere - which we have already derived as equation 4.

Finally, for compatibility with Susskind we want the equation of motion by calculating the acceleration 'a'. This is done using 'a = F / m'. Giving :-

$$a = \frac{4\pi G R \rho}{3} \qquad \text{Equation 5}$$

Susskind does thing it terms of a scale factor, for which he confusing uses 'a', which effectively replaces 'R'.
Thus the acceleration will be directly towards the observer and the magnitude will be proportional to the distance from the observer to the test mass element. This suggests that the universe will collapse towards the observer

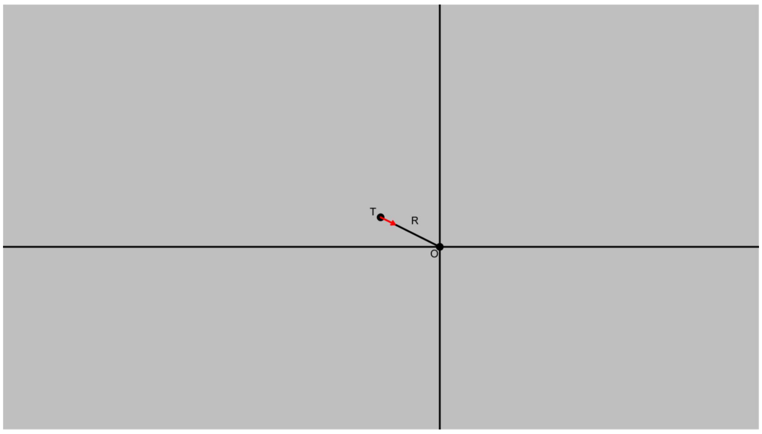


Figure 6: Acceleration Towards Observer

What is the Problem?
All the above physics is correct, so why do I dispute the conclusion? First let us consider the system from the point of view of another observer in the same inertial reference frame.

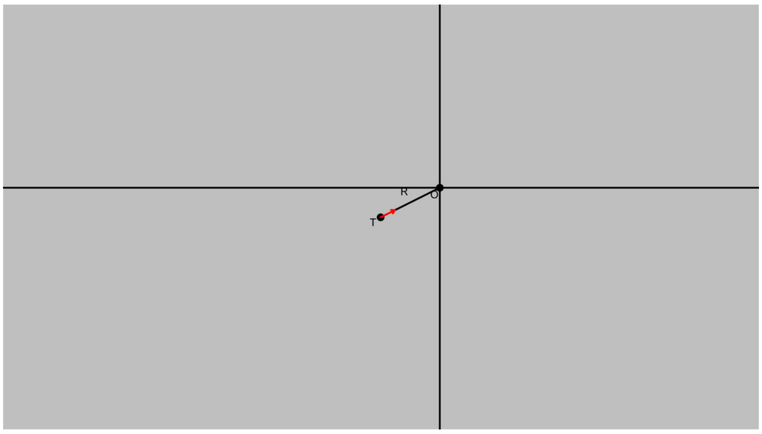


Figure 7: Acceleration Towards Another Observer

The test particle is now being pulled in a totally different direction. We could keep doing this for any number of observers and they will all deduce that the test particle should be accelerating towards them. When one of the students challenged Susskind with an argument like this he dismissed to objection saying that changing the observer would introduce pseudo-forces, but this would only arise in the second observe were accelerating – which he is not.

At this point you may be thinking of the the Hubble expansion of the universe, where all observers see themselves as the centre of expansion. But in that case each observer is moving relative to the other observers and the reference frame is moving with them. In our case the different observes are all static relative to each other. Also there is no reason for them to be moving. If we set the system up to be initially at rest then even if there were an acceleration it would take time for the speed to build up.

Where did the Analysis go Wrong?
The error in the reasoning is the infinite sum for the shells. We can consider a shell as two hemispheres pulling in opposite directions with equal force. The hemispheres for all shells will be the same – the mass will be proportional to R squared but the inverse square law will cancel this out. We can look at the final sum (with appropriate scaling) as :-

1 – 1 + 1 – 1 + 1 – 1 + 1 – 1 + 1 – 1 + 1 – 1 . . .

We can group terms as follows :-

(1 – 1) + (1 – 1) + (1 – 1) + (1 – 1) + (1 – 1) + (1 – 1) . . .

This will reduce to :-

0 + 0 + 0 + 0 + 0 + 0 . . .

Which will give zero.

However we could also group terms as :-

1 + (– 1 + 1) + (– 1 + 1) + (– 1 + 1) + (– 1 + 1) + (– 1 + 1) + (– 1 . . .

This will reduce to :-

1 + 0 + 0 + 0 + 0 + 0 . . .

Which will equal one.

You could also start by picking out any number of 1s then start pairing all the -1s in turn with all the unmatched 1s. This will yield any number you like.

This is effectively what you are doing when you extract a core of the space then start adding shells that are asymmetric relative to the target mass.

If you watch Guth's lecture from time 0:59:00 you will see another way of explaining the discrepancy.

The Gauss's Law Approach

In Guth's lecture, at time 0:48:00, an alternative approach is introduced. This involves Gauss's Law of Gravitation, which is analogous to his law of electrostatics.

Gauss's Law of Gravitation can be expressed as follows :-

$$\oint_S \overline{g} \cdot \overline{dA} = 4\pi GM \qquad \text{Equation 6}$$

This states that the net gravitational flux over the surface of a region of space depends only upon the matter within that region. Any matter outside the region will contribute to the flux flowing into the region at some point but this will be balanced by extra flux flowing out of the region at another point. Here I am using 'gravitational flux' as dot product of field strength and area, in analogy to electric flux. I am not going to go into the details here as there are multiple resources on the internet that can explain it far better than I can.

As any volume of space contains some matter then there must be a non zero net flux into the region.

So there must be some non-zero gravitational field at some point. Thus there must be a force on some matter, meaning that the system is not perfectly gravitationally balanced.

What is the Problem?
Again it is relatively easy to show that this does not work but harder to show where the error has occurred.

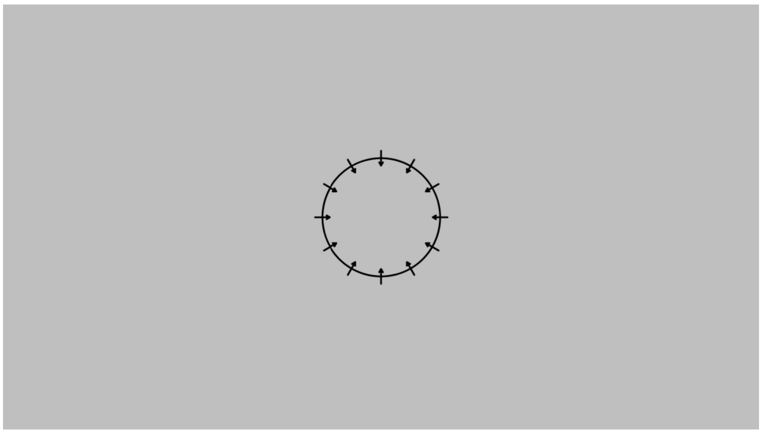
If we choose some arbitrary point in space as the centre of a sphere of radius R then the total integral for the flux will be the gravitational acceleration times the surface area of the sphere :-

$$-g4\pi R^2 = 4\pi G\frac{4}{3}\pi R^3\rho \qquad \text{Equation 7}$$

This will reduce to :-

$$g = \frac{-4\pi G R\rho}{3} \qquad \text{Equation 8}$$

From symmetry we see that this must be uniformly spread around the surface and pointing towards the centre of the sphere (the external contributions around the surface will also be symmetrical and thus be zero at all points – because the total contribution will be zero).



1. *Figure 8: Field over Gaussian Surface*

If we now choose another sphere that touches the first sphere but does not intersect then field at the surface of this second sphere must be in the opposite direction and have a magnitude determined by the radius of the second sphere. The field at a given point cannot have two different magnitudes and directions.

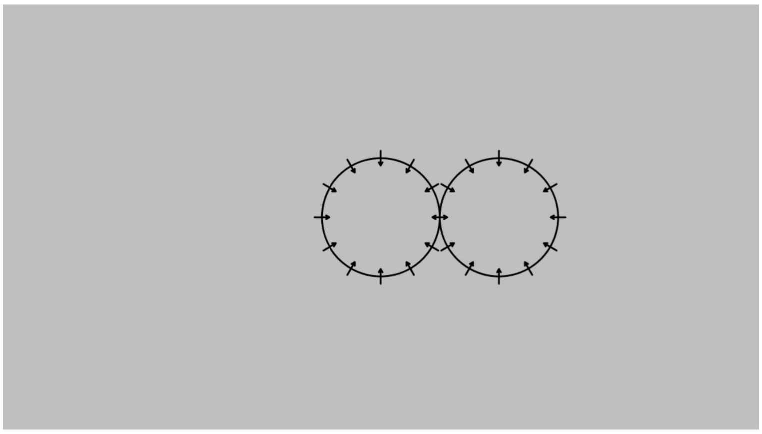


Figure 9: Field over Touching Gaussian Surfaces

Another way to look at it is in terms of homogeneity. If there is a field it must be the same strength and direction everywhere thus the whole universe would move in the same direction rather than collapsing.

Where did the Analysis go Wrong?

To understand my idea about why the Gaussian approach failed we need to go back to the original electrostatic version of Gauss's Law. In electrostatics there are two types of charge – positive and negative. Any line of an electric field starts at a positive charge and ends at a negative charge.

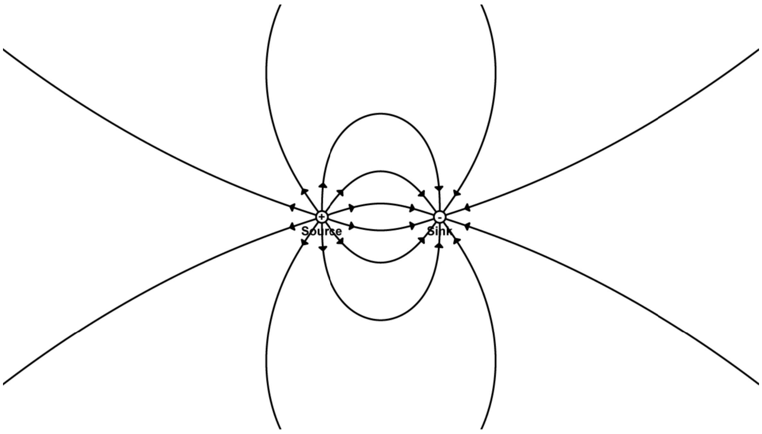


Figure 10: Typical Electric Field Lines

Any line in the above diagram that disappears on the left will reappear on the right. The universe seems to be more or less electrically neutral so all sources of electric field have a sink. Thus all field lines can go somewhere. When we extend the approach to the gravitational field there are no “negative masses” to act as sinks for the field. This is not a problem if the matter field of the universe is finite and the field lines can go off to an infinite distance beyond any matter and be quietly forgotten.

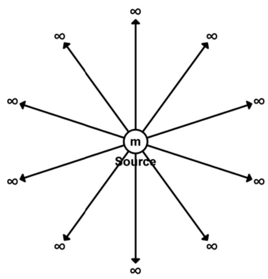


Figure 11: Gravitational Field

The trouble is when the matter field is infinite and there is nowhere to act as a sink for the field lines. Thus the field lines generated within one of the enclosed regions of space cannot escape and hence the integral of the flux out must on average be zero and for a uniform matter distribution it will always be zero for any closed surface. So there would be no Gauss's Law of Gravity in an infinite Newtonian Universe. It is also interesting to consider an infinite universe and removing all the negative charges: would the electrostatic version of Gauss's law still hold?

Laplace's Equation Approach

This is the alternative approach that Guth introduces in his lecture at timestamp 0:54:40. He gives the equation as :-

$$\nabla^2\phi = 4\pi G\rho \qquad \text{Equation 9}$$

Some sources call the the Poisson Equation of the Poisson-Laplace equation (in this case I think Poisson-Laplace would be appropriate given how fishy I find Guth's analysis).

In the above the left hand side expands as shown :-

$$\nabla^2\phi = \frac{\partial\phi^2}{\partial x^2} + \frac{\partial\phi^2}{\partial y^2} + \frac{\partial\phi^2}{\partial z^2} \qquad \text{Equation 10}$$

In these equations the symbol ϕ is the gravitational potential.

Guth's arguments is that if ϕ is constant everywhere then the second derivative of the potential is zero everywhere and in all directions. Thus the Laplacian would be zero. The Laplacian being zero would mean that the mass density must be zero. So an infinite universe with a non-zero mass density must have a potential that varies, and as the gradient of the potential gives the net gravitational force at that point this means that there must be a net gravitational force everywhere.

What is the Problem?

The underlying specification of the uniform universe implies that there is no variation in gravitational potential as any variation would violate the assumption of homogeneity. If there were a gradient in the potential field this would be a vector and thus have a direction, thus violating isotropy.

Where did the Analysis go Wrong?
The problem is once again the lack of empty space outside the matter field. For a finite universe the zero potential is at a point infinitely far from the matter, any all potentials are negative relative to this. There is a conflict between the Laplace's Equation and the uniformity of the potential field, so Something must give. Guth assumes that the uniformity needs to go, I believe it is Laplace's equation that must be abandoned. Laplace's Equation is derived from the underlying laws of gravity with some implicit assumptions – such as a meaningful zero point far away from the matter. This assumption does not hold for an infinite matter field.

Conclusion
So, no matter which approach you take, I still think that Newton was correct in his assumptions about the stability of his universe – for the ideal case. In real life the uniformity will not be perfect, so there will be local contractions, but not a uniform global contraction.